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Technical Note

Stability of natural convection in superposed fluid and porous layers: Equivalence of the one- and two-domain approaches

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ABSTRACT

Stability analyses of thermal and/or solutal natural convection in a configuration composed by a fluid layer overlying a homogeneous porous medium have been performed using different modeling approaches, especially for the treatment of the interfacial region. Comparisons between the one-domain approach and the two-domain formulation have shown important discrepancies of the marginal stability curves. This note shows that, according to Kataoka [I. Kataoka, Local instant formulation of two-phase flow, Int. J. Multiphase Flow 12(5) (1986) 745–758.], the differentiation of the macroscopic properties of the porous layer at the interface (porosity, permeability, thermal effective diffusivity) must be considered in the meaning of distributions. In that case, the one- and the two-domain approaches are shown to be equivalent and very good agreement is indeed found when comparing the results obtained with both approaches.

1. Introduction

Due to its fundamental and practical interests, the stability analysis of thermal and/or solutal natural convection in a configuration composed by a fluid layer overlying a homogeneous porous medium has been the subject of particular attention since the pioneering study performed by Nield [2]. In most cases, linear stability analyses for the onset of natural convection in this configuration have been performed using a two-domain approach (2Ω) where conservation equations in the fluid and porous regions are coupled by interfacial boundary conditions.

For momentum transport, the large majority of the studies [2–6] uses Darcy's law in the porous region ($2\Omega_D$) and the coupling with the Navier–Stokes equations in the fluid region is performed using a slip boundary condition [7] involving a dimensionless adjustable slip coefficient α that depends on the local geometry of the interface [8]. An alternative two-domain representation consists in using the Darcy–Brinkman equation [9–11] allowing to satisfy continuity of both velocity and shear stress at the fluid/porous interface ($2\Omega_{DB}$). The comparison between the marginal stability curves obtained with ($2\Omega_D$) and ($2\Omega_{DB}$) are roughly in good agreement for $1 \leqslant \alpha \leqslant 4$, whatever the depth ratio ($\hat{d} = d_f/d_m$) (d_f and d_m being the thicknesses of the fluid and porous layers, respectively) [12]. The stability curves can present a bimodal behaviour depending on the values of the characteristic parameters. In the one-domain approach (1Ω), the porous layer is viewed as a pseudo-fluid and the whole cavity as a continuum [13]. In that case, heat and/or mass transfer are governed by a unique set of conservation equations both valid in the fluid and porous regions avoiding the explicit formulation of the boundary conditions at the interface. Very few stability analyses have been performed using (1Ω) and the comparison with the results obtained with ($2\Omega_D$) or ($2\Omega_{DB}$) shows important discrepancies [4,14,12].

The objective of this note is to show that the one-domain approach is actually equivalent to the ($2\Omega_{DB}$). Indeed, in this formulation the average properties of the porous media (porosity, permeability, effective diffusivity,...) are Heaviside functions and therefore their differentiation must be performed in the meaning of distributions [1,15]. The linear stability analysis is developed and the eigenvalue problem is solved using the Generalized Integral Transform Technique (GITT) [16]. The comparison of the marginal stability curves shows, for different values of the depth radio, a perfect agreement between the one- and the two-domain formulations.

2. Mathematical modeling

The geometrical configuration is composed by an infinite horizontal porous layer of thickness d_m^* underlying a fluid layer of thickness $d_f^* = d^* - d_m^*$ (Fig. 1). The porous layer is assumed to be *homogeneous*, isotropic and saturated by the overlying fluid which is assumed to be Newtonian and to satisfy the linear Boussinesq approximation $\rho(T) = \rho_0(1 - \beta_T(T - T_0))$ where ρ_0 represents the

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Nomenclature

d^*	total thickness, m
$\hat{d}$	depth ratio ($\hat{d} = d_{\rm f}^*/d_{\rm m}^*$)
Da	Darcy number ($Da = K/d^{*2}$)
g	gravity constant, m s^{-2}
Gr_T	Grashof number ($Gr_T = g\,\beta_T \Delta T d^{*3}/\nu^2$)
K	permeability of the porous medium, m^2
k	thermal conductivity, Wm^{-1}K^{-1}
Pr	Prandtl number ($Pr = \nu/\alpha_{T{\rm f}}$)
Ra_T	Rayleigh number Ra_T=Gr^T Pr Da)
T	temperature field, K
$T_{\rm b}$	temperature at the bottom boundary, K
$T_{\rm u}$	temperature of the upper boundary, K
$\mathbf{u}$	velocity field, m s^{-1}
w	vertical velocity component, m s^{-1}

Greek symbols

α	slip coefficient
α_T	thermal diffusivity, m^2 s^{-1}
β_T	thermal expansion coefficient, K^{-1}
κ	dimensionless wave number
μ	dynamic viscosity of the fluid, kg m^{-1} s^{-1}
ν	kinematic viscosity of the fluid, m^{2}s^{-1}
ρ	fluid density, kg m^{-3}
σ	growth rate
ϕ	porosity

Subscripts

0	reference
f	fluid property
m	porous medium property

density at the temperature T_0. The horizontal walls are impermeable and are maintained at temperatures $T_{\rm u}$ (top) and $T_{\rm b}$ (bottom). The dimensionless mass, momentum and energy conservation equations are obtained using the following scales: d^* for length, d^{*2}/ν for time, ν/d^* for velocity, and $(\rho_0\nu^2)/d^{*2}$ for pressure, ν being the kinematic viscosity. The temperature difference $(T - T_0)$ is scaled by $\Delta T = T_{\rm u}\text{-}T_{\rm b}$. Therefore, the set of equations can be written as

$$\nabla\cdot\mathbf{u} = 0 \tag{1}$$

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{u}}{\phi}\right) + \frac{1}{\phi}\left(\mathbf{u}\cdot\nabla\frac{\mathbf{u}}{\phi}\right) = \nabla\cdot\left(\frac{1}{\phi}\nabla\mathbf{u} - P\mathbf{I}\right) - \frac{1}{Da}\mathbf{u} + Gr_T T\mathbf{e}_z \tag{2}$$

$$\alpha_{T{\rm f}}\left(\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T\right) = \frac{1}{Pr_{\rm f}}\nabla\cdot(\alpha_T\nabla T) \tag{3}$$

where $Gr_T = (\rho_0 g\beta_T\Delta T\, d^3)/\nu^2$ is the thermal Grashof number based on the total depth of the channel d and $Pr_{\rm f}$ is the fluid Prandtl number. In Eqs. (1)–(3) ϕ represents the porosity, $Da = K/d^{*2}$ the dimensionless permeability and α_T is the thermal diffusivity ($\alpha_{T{\rm f}}$ being the thermal diffusivity in the fluid). In the momentum Eq. (2) the reduced viscosity in the Brinkman term has been taken such as $\eta = \mu_{\rm eff}/\mu_{\rm f} = 1/\phi$ [17]. In this equation, the second Brinkman correction term has been neglected. The dimensionless boundary conditions at the external walls take the form

$$\mathbf{u}(1) = \mathbf{u}(0) = 0, \quad T(1) = \frac{T_{\rm u}^* - T_0^*}{\Delta T^*}, \quad T(0) = \frac{T_{\rm b}^* - T_0^*}{\Delta T^*} \tag{4}$$

In this one-domain formulation, the effective properties (ϕ, Da, and α_T) are discontinuous functions and therefore their differentiation must be considered in the meaning of the distribution [1,15].

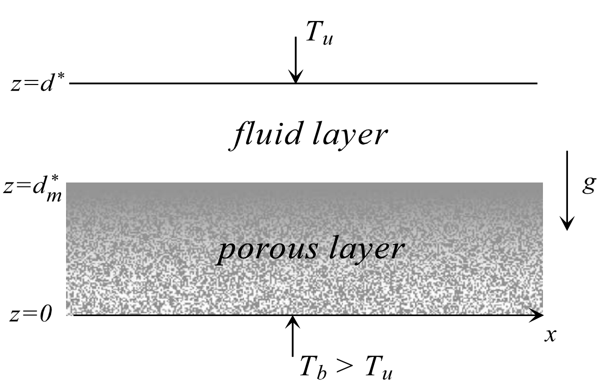

Fig. 1. Geometrical configuration.

Before performing the linear stability analysis using the one-domain formulation, let us recall the dimensionless form of the two-domain approach using the Darcy–Brinkman equation in the porous region [18]. The conservation equations for the fluid layer are

$$\nabla\cdot\mathbf{u}_{\rm f} = 0 \tag{5}$$

$$\frac{\partial\mathbf{u}_{\rm f}}{\partial t} + \mathbf{u}_{\rm f}\cdot\nabla\mathbf{u}_{\rm f} = -\nabla P_{\rm f} + \nabla^2\mathbf{u}_{\rm f} + Gr_{T{\rm f}} T\mathbf{e}_z \tag{6}$$

$$\frac{\partial T_{\rm f}}{\partial t} + \mathbf{u}_{\rm f}\cdot\nabla T_{\rm f} = \frac{1}{Pr_{\rm f}}\nabla^2 T_{\rm f} \tag{7}$$

while the dimensionless equations for the porous medium are given by

$$\nabla\cdot\mathbf{u}_{\rm m} = 0 \tag{8}$$

$$\frac{1}{\phi}\frac{\partial\mathbf{u}_{\rm m}}{\partial t} = -\nabla P_{\rm m} + \frac{1}{Da}\mathbf{u}_{\rm m} + \eta\nabla^2\mathbf{u}_{\rm m} + Gr_T T_{\rm m}\mathbf{e}_z \tag{9}$$

$$\sigma_{\rm m}\frac{\partial T_{\rm m}}{\partial t} + \mathbf{u}_{\rm m}\cdot\nabla T_{\rm m} = \frac{1}{Pr_{\rm m}}\nabla^2 T_{\rm m} \tag{10}$$

where $Pr_{\rm m}$ represents an effective Prandtl number for the porous region ($Pr_{\rm m} = \nu/\alpha_{T_{\rm m}}$) and $\sigma_{\rm m} = (\rho_0 C_{\rm pm})/(\rho_0 C_{\rm pf})$. The dimensionless boundary conditions at the top and bottom walls remain unchanged while at the interface ($z = d_{\rm m} = d_{\rm m}^*/d^* = 1/(1+\hat{d})$ where $\hat{d} = d_f^*/d_{\rm m}^*$) the dimensionless boundary conditions take the form

$$T_{\rm f} = T_{\rm m} \tag{11}$$

$$\frac{\partial T_{\rm f}}{\partial z} = \frac{1}{\varepsilon_T}\frac{\partial T_{\rm m}}{\partial z} \tag{12}$$

$$\mathbf{u}_{\rm f} = \mathbf{u}_{\rm m} \tag{13}$$

$$-P_{\rm f} + 2\frac{\partial w_{\rm f}}{\partial z} = -P_{\rm m} + \frac{2}{\phi}\frac{\partial w_{\rm m}}{\partial z} \tag{14}$$

$$\frac{\partial\mathbf{u}_{\rm f}}{\partial z} = \frac{1}{\phi}\frac{\partial\mathbf{u}_{\rm m}}{\partial z} \tag{15}$$

where $\varepsilon_T = \alpha_{T_{\rm f}}/\alpha_{T_{\rm m}}$.

3. Linear stability analysis

For the sake of conciseness, only the stability analysis for the one-domain approach given by Eqs. (1)–(4) is presented in this section. The corresponding analysis for the two-domain Darcy–Brinkman formulation is detailed in Hirata et al. [12]. The perturbation equations are obtained in a usual way using the general decomposition

$$\zeta = \bar{\zeta}(z) + \zeta'(x,z,t) \tag{16}$$

where the overlined and prime notations represent the basic state and the perturbation of a generic variable ζ, respectively. The basic state is assumed to be quiescent and therefore the velocity components are such as $\bar{u}(z)=\bar{w}(z)=0$ and $\partial/\partial t=0$. Eq. (16) is introduced in Eqs. (1)–(3) and the resulting system is linearized. In order to eliminate the pressure term, Eq. (2) is operated with $(\nabla\times\nabla\times)$ and applying continuity, the z-component of the momentum equation takes the form

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}\left(-\frac{1}{\phi}\frac{\partial w'}{\partial z}\right)+\frac{1}{\phi}\left(\frac{\partial^2 w'}{\partial x^2}\right)\right)=\frac{\partial}{\partial z}\left(\frac{1}{Da}\right)\frac{\partial w'}{\partial z}+\frac{1}{Da}\frac{\partial^2 w'}{\partial x^2}+\frac{1}{Da}\frac{\partial^2 w'}{\partial z^2}\\&-\frac{1}{\phi}\nabla^2\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right)-\frac{\partial}{\partial z}\left(\frac{1}{\phi}\right)\nabla^2\left(\frac{\partial w'}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{1}{\phi}\right)\frac{\partial^3 w'}{\partial z^3}-\frac{\partial^2}{\partial z^2}\left(\frac{1}{\phi}\right)\frac{\partial^2 w'}{\partial z^2}\\&-\frac{\partial}{\partial z}\left(\frac{1}{\phi}\right)\frac{\partial^3 w'}{\partial x^2\partial z}+Gr_T\frac{\partial^2 T'}{\partial x^2}\end{aligned}\tag{17}$$

Similarly, the linearized energy Eq. (3) takes the form

$$Pr_f\alpha_{Tf}\left(\frac{\partial T'}{\partial t}+w'\frac{\partial \overline{T}}{\partial z}\right)=\alpha_T\left(\frac{\partial^2 T'}{\partial x^2}+\frac{\partial^2 T'}{\partial z^2}\right)+\frac{\partial \alpha_T}{\partial z}\frac{\partial T'}{\partial z}\tag{18}$$

According to the normal mode expansion, the vertical velocity component and the temperature are decomposed under the form

$$(w',T')=(W(z),\theta(z))\mathbf{e}^{i\kappa x+\sigma t}\tag{19}$$

with $\nabla_2^2 f+\kappa^2 f=0$ ($\nabla_2^2=\partial^2/\partial x^2$) and where $W(z)$ and $\theta(z)$ are the amplitude of the velocity and temperature, respectively. κ is the dimensionless wave number and σ is the complex growth rate. Assuming that the principle of exchange of stability holds ($\sigma=0$) and introducing Eq. (19) into the linearized system gives

$$\begin{aligned}&\frac{1}{\phi}\left(\frac{d^4W}{dz^4}+\kappa^4 W\right)-\kappa^2\left(\frac{2}{\phi}\frac{d^2W}{dz^2}-\frac{1}{Da}W+2\frac{d}{dz}\left(\frac{1}{\phi}\right)\frac{dW}{dz}\right)\\&-\left(-2\frac{d}{dz}\left(\frac{1}{\phi}\right)\frac{d^3W}{dz^3}+\frac{d}{dz}\left(\frac{1}{Da}\right)\frac{dW}{dz}+\frac{1}{Da}\frac{d^2W}{dz^2}-\frac{d^2}{dz^2}\left(\frac{1}{\phi}\right)\frac{d^2W}{dz^2}\right)+\kappa^2 Gr_T\theta=0\end{aligned}\tag{20}$$

$$Pr_f\alpha_{Tf}\left(\frac{d\overline{T}}{dz}W\right)=\alpha_T\left(-\kappa^2\theta+\frac{d^2\theta}{dz^2}\right)+\frac{d\alpha_T}{dz}\frac{d\theta}{dz}\tag{21}$$

The system (20 and 21), representing the eigenvalue problem is solved using the Generalized Integral Transform Technique (GITT) [16] and the critical Grashof number and the critical wave number are determined by minimization over κ. For conciseness, the GITT is not described in the present note and details concerning its application to such a fluid-porous configuration can be found in Hirata et al. [12,19]. This numerical method has been validated by comparison with the exact values obtained in full fluid and porous cavities [20].

As previously said ϕ, Da and α_T are Heaviside functions and their differentiation in Eqs. (20) and (21) must be performed in the sense of distributions. In order to be more explicit, let us consider the last term of the RHS of Eq. (21) and introduce the porous region indicator function γ defined by

$$\gamma(z)=\begin{cases}1 & \text{for}\quad z<d^*_m\\ 0 & \text{for}\quad z>d^*_m\end{cases}\tag{22}$$

Therefore, α_T can be written under the form

$$\alpha_T(z)=\gamma(z)\alpha_{T_m}(z)+(1-\gamma(z))\alpha_{T_f}(z)\tag{23}$$

where $\alpha_{T_m}(z)$ and $\alpha_{T_f}(z)$ are regular functions. Since $\gamma(z)$ is a Heaviside function, its derivative in the sense of distribution takes the form

$$\frac{d\gamma}{dz}=-\delta_{\text{int}}(z-d^*)\tag{24}$$

where $\delta_{\text{int}}(z-d^*)$ is the Dirac delta function at the interface. Under these circumstances

$$\frac{d\alpha_T}{dz}=\gamma(z)\frac{d\alpha_{T_m}}{dz}+(1-\gamma(z))\frac{d\alpha_{T_f}}{dz}-(\alpha_{T_m}-\alpha_{T_f})\delta_{\text{int}}\tag{25}$$

Since the fluid and the porous regions are both considered homogeneous (i.e., $d\alpha_{T_m}/dz=0=d\alpha_{T_f}/dz$), the last term of the RHS of Eq. (21) takes the form

$$\frac{d\alpha_T}{dz}\frac{d\theta}{dz}=(\alpha_{T_f}-\alpha_{T_m})\delta_{\text{int}}\frac{d\theta}{dz}\tag{26}$$

The numerical treatment of the Dirac delta contributions in the GITT is detailled in Hirata (2007) [21].

4. Results and conclusions

The comparison of the marginal stability curves obtained with the two-domain approach ($2\Omega_{DB}$) and the present one-domain formulation using the differentiation of the discontinuous functions in the meaning of distribution are presented in Fig. 2. The marginal stability curves represent the thermal Rayleigh number Ra_T versus

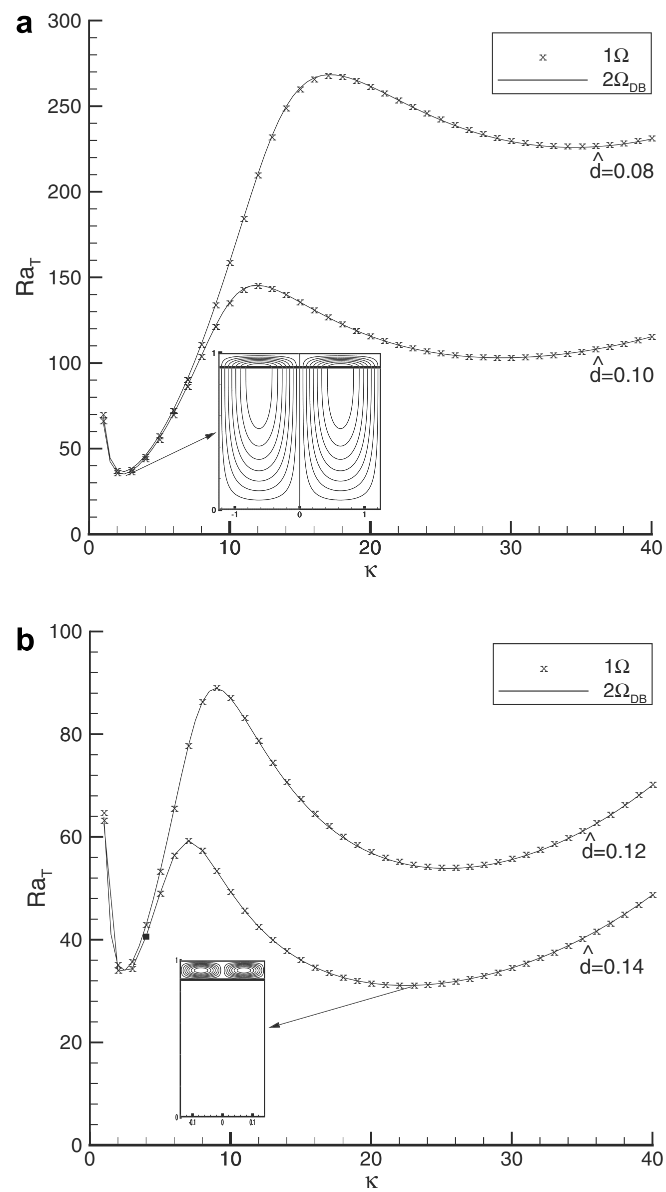

Fig. 2. Marginal stability curves: comparison between the one-domain (1Ω) and the two-domain ($2\Omega_{DB}$) approaches for (a) $\hat{d}=d^*_f/d^*_m=0.08$ and $\hat{d}=0.10$; (b) $\hat{d}=0.12$ and $\hat{d}=0.14$. The Darcy number is fixed: $Da = 7.44\ 10^{-6}$.

the wave number κ for different values of the depth ratio ($\hat{d}$). It is shown that perfect agreement is obtained between the one-domain (1Ω) and the two-domain ($2\Omega_{DB}$) approaches. Indeed, both approaches present exactly the same bimodal behavior whatever the thickness ratio $\hat{d}$. Each mode corresponds to a different structure of the convective flow. Indeed, at small wave numbers ($\kappa \sim 2.5$) the convective flow occurs in the whole cavity (“porous mode”) while perturbations of large wave numbers lead to a convective flow mainly confined in the fluid layer (“fluid mode”). In (Fig. 2b), a transition between the porous mode and the fluid mode is observed at $\hat{d} = 0.14$.

In conclusion, this analysis shows that the one- and two-domain approaches are equivalent, provided that the one-domain approach is properly interpreted mathematically, i.e. in the meaning of distributions. The generalization of the equivalence between the one- and two-domain approaches for the modeling of transport phenomena at a fluid/porous interface is under development.

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